Adic and perfectoid spaces

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Abstract

The main goal of this course is to develop the foundations of the theory of perfectoid spaces (P), more precisely to prove the various tilting correspondences for perfectoid rings, the almost purity theorem, and almost vanishing theorems. We develop simultaneously what is needed from the theory of adic spaces.

Contents

Some notation

For the whole course we fix a prime number p. All rings are commutative. Given a ring A of characteristic p, we write $\varphi : A \to A$, $a \mapsto a^p$ for its Frobenius endomorphism (which is a ring homomorphism).

1 (P) Integral perfectoid rings

Let A be a topological ring. We say that A is *integral perfectoid* if and only if there exists a non-zero-divisor $\pi \in A$ such that

(a) the topology on A is the π-adic topology, and A is complete for this topology (i.e., $A \to \varprojlim_n A/\pi^n A$ is an isomorphism of topological rings, where each $A/\pi^n A$ has the discrete topology);

(b) $p \in \pi^p A$;

(c) $\Phi : A/\pi A \to A/\pi^p A$, $a \mapsto a^p$, is an isomorphism

It is convenient for us, even though it is not exactly standard in the literature, to call any such element π a perfectoid pseudo-uniformiser (ppu).

Example 1.1. Here are some easy examples:

(i) The p-adic completions of the rings $\mathbb{Z}_p[p^{1/p^\infty}]$ and $\mathbb{Z}_p[\zeta_{p^\infty}]$ are perfectoid (equipped with the p-adic topology), with perfectoid pseudo-uniformisers $p^{1/p}$ and $\zeta_{p^2} - 1$ respectively.

(ii) $\mathbb{Z}_p$ is not perfectoid.

Construction 1.2. Given any ring A we write

$$\varprojlim_{x \mapsto x^p} A := \{(a_0, a_1, \dots) \in A^{\mathbb{N}} : a_i^p = a_{i-1} \text{ for all } i \geq 1\}$$

for the set of compatible sequences of p-power roots in A. Note that we can multiply two such sequences, so $\varprojlim_{x\mapsto x^p} A$ forms a multiplicative monoid; if A has characteristic p then we can also add two such sequences, so then A is even a ring.

The construction is clearly functorial: a morphism of rings $A \to B$ induces a morphism of monoids $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} B$ (which is even a morphism of rings if A and B have characteristic p).

The following result is extremely important and will be used repeatedly: if $\pi \in A$ is an element such that (i) $p \in \pi A$ and (ii) A is π-adically complete, then the map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/\pi A$ is actually a bijection (hence an isomorphism of monoids). We leave the details of the proof to the reader, but give the following recipe for the inverse of the map: given $b = (b_0, b_1, \dots) \in \varprojlim_{x\mapsto x^p} A/\pi A$, let $\widetilde{b_i} \in A$ be an arbitrary lift of $b_i \in A/\pi A$ for each $i \geq 0$. Then set

$$a_i := \lim_{n\to\infty} \widetilde{b_{i+n}}^{p^n}$$

and check that $a := (a_0, a_1, \dots) \in \varprojlim_{x\mapsto x^p} A$ really is a well-defined lift of b.

Lemma 1.3. *Let A be integral perfectoid, and $\pi \in A$ a perfectoid pseudo-uniformiser. Then:*

(i) Every element of $A/\pi p A$ is a p^{th}-power (n.b., $A/\pi p A$ does not necessarily have characteristic p).

(ii) If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$, then $a \in A$.

(iii) After multiplying π by a unit it admits a compatible sequence of p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots \in A$.

Proof. (i): Using the surjectivity of Φ, a simple induction lets us write any $a \in A$ as an infinite sum $a = \sum_{i\geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$; but this is $\equiv (\sum_{i\geq 0} a_i \pi^i)^p \bmod p\pi A$.

(ii): Let $l \geq 0$ be the smallest integer such that $\pi^l a \in A$. Assuming that $l > 0$, we get a contradiction by noting that $\pi^{pl} a^p \in \pi^{pl} A \subseteq \pi^p A$, whence $\pi^l a \in \pi A$ by condition (c), and so $\pi^{l-1} a \in A$.

(iii): Since the Frobenius is surjective on $A/\pi^p A$, there exists an element of $\varprojlim_{x\mapsto x^p} A/\pi^p A$ of the form $(\pi \bmod \pi^p A, ?, ?, \dots)$. Applying the exercise of Construction 1.2, we deduce that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/\pi^p A$ is a bijection. Hence there exists $a = (a_0, a_1, \dots) \in \varprojlim_{x\mapsto x^p} A$ such that $a_0 \equiv \pi \bmod \pi^p A$; therefore $a = u\pi$ for some $u \in 1 + \pi^{p-1} A \subseteq A^\times$ (the inclusion $\subseteq$ results from π-adic completeness of A). $\square$

Lemma 1.4. *Let A be integral perfectoid, and $\varpi \in A$ an element satisfying conditions (a) and (b). Then ϖ is a non-zero-divisor satisfying (c), i.e., it is a perfectoid pseudo-uniformiser.*

Proof. We must show that $\Phi : A/\varpi A \to A/\varpi^p A$ is an isomorphism. Let $\pi \in A$ be a perfectoid pseudo-uniformiser.

It follows from Lemma 1.3(i) that every element of A/pA is a p^{th}-power; hence every element of its quotient $A/\varpi^p A$ is a p^{th}-power, i.e., Φ is surjective.

The fact that π and ϖ define the same topology implies that a power of each is divisible by the other, whence ϖ is a non-zero-divisor and $A[\frac{1}{\varpi}] = A[\frac{1}{\pi}]$. If $a \in A$ satisfies $a^p \in \varpi^p A$, then $(a/\varpi) \in A[\frac{1}{\pi}]$ satisfies $(a/\varpi)^p \in A$, and it then follows from Lemma 1.3(ii) that in fact $a \in \varpi A$ as desired. $\square$

Lemma 1.5. *Suppose A is a complete topological ring such that $pA = 0$. Then A is integral perfectoid if and only if it is perfect and the topology is π-adic for some non-zero-divisor $\pi \in A$.*

Proof. Exercise. $\square$

1.1 The tilt of an integral perfectoid ring

Definition 1.6. The *tilt* of an integral perfectoid ring A is $A^\flat := \varprojlim_{x\mapsto x^p} A/pA$, equipped with the inverse limit topology (A/pA is of course given the quotient topology, i.e., the π-adic topology for any choice of pup for A).

Note that $A^\flat$ is a perfect ring of characteristic p; in fact, it is the initial object among all perfect rings of characteristic p mapping to A/pA.

Recalling from Construction 1.2 that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_\varphi A/pA$ is an isomorphism of monoids, we define the *untilting map* $\# : A^\flat \to A$, $b \mapsto b^\#$ to be projection to the 0^{th}-coordinate of $\varprojlim_{x\mapsto x^p} A$; explicitly, the map $\#$ is given by $\varprojlim_\varphi A/pA \ni (b_0, b_1, \dots) \mapsto \lim_{i\to\infty} \widetilde{b_i}^{p^i}$, where $\widetilde{b_i} \in A$ are arbitrary lifts of the elements $b_i \in A/pA$.

The untilting map is multiplicative by generally not additive; in fact, given $b, c \in A^\flat$, it transforms under addition as follows:

$$(b + c)^\# = \lim_{i\to\infty} ((b^{1/p^i})^\# + (c^{1/p^i})^\#)^{p^i}.$$

However, note that the composition $A^\flat \overset{\#}{\longrightarrow} \overset{\bmod p}{\longrightarrow} A/pA$ is a ring homomorphism: indeed, it is the surjective ring homomorphism given by projecting $A^\flat \cong \varprojlim_\varphi A/pA$ to the 0^{th}-coordinate. Also, if A is of characteristic p, then the untilting map $\# : A^\flat \to A$ is an isomorphism of rings.

Lemma 1.7. *Let A be an integral perfectoid ring. Then:*

(i) $\# : A^\flat \to A$*, is continuous;*

(ii) the isomorphisms of monoids $\varprojlim_{x\mapsto x^p} A \to A^\flat = \varprojlim_{x\mapsto x^p} A/pA \to \varprojlim_{x\mapsto x^p} A/\pi A$ *are homeomorphisms, where* $\pi \in A$ *is any perfectoid pseudo-uniformiser;*

(iii) $A^\flat$ *is also an integral perfectoid ring.*

Proof. Given $(1,\dots,1,b_{n+1},b_{n+2},\dots) \in \varprojlim_\varphi A/\pi A$, any chosen lifts $\widetilde{b_i}$ satisfy $\widetilde{b_i}^{p^{i-n}} \equiv 1 \bmod \pi A$ for $i > n$, whence $\widetilde{b_i}^{p^i} \equiv 1 \bmod \pi^n A$; taking the limit shows that the untilt is $\equiv 1 \mod \pi^n A$. This proves that the untilting map $\# : \varprojlim_{x\mapsto x^p} A/\pi A \to A$ is continuous (for the inverse limit of discrete topologies on the domain), from which (i) and (ii) easily follow. Filling in the details is left as an exercise.

(iii) We have already noted that $A^\flat$ is a perfect ring of characteristic, and the homeomorphism $A \cong \varprojlim_{x\mapsto x^p} A/\pi A$ shows that A is an inverse limit of discrete rings, whence A is a complete topological ring. According to Lemma 1.5, it remains to prove the following: there exists a non-zero-divisor $\pi^\flat \in A^\flat$ such that the topology on $A^\flat$ is the $\pi^\flat$-adic topology.

Possibly after changing our perfectoid pseudo-uniformiser π, we may assume that it admits compatible p-power roots (by Lemma 1.3(iii)); let $\pi^\flat = (\pi, \pi^{1/p},\dots,) \in A^\flat$ be the corresponding element of $A^\flat$, which satisfies $(\pi^\flat)^\# = \pi$.

We show first that $\pi^\flat$ is a non-zero-divisor. To do that we note that for each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow \pi^{1-1/p^n} A/\pi A \longrightarrow A/\pi A \overset{\times \pi^{1/p^n}}{\longrightarrow} A/\pi A \overset{\varphi^n}{\longrightarrow} A/\pi A \longrightarrow 0$$

Exactness is easy everywhere except possibly at the second term from the right: but if $a \in A$ satisfies $a^p \in \pi A$ then $a/\pi^{1/p^n} \in A[\frac{1}{\pi}]$ satisfies $(a/\pi^{1/p^n})^{p^n} \in A$, whence Lemma 1.3(ii) implies $a \in \pi^{1/p^n} A$ as desired.

These sequences are moreover compatible in n, with respect to the maps $0, \varphi, \varphi, \mathrm{id}$ respectively. Although it is not always the case that an inverse limit of exact sequences is still exact, in this case the transition map are either surjective (φ and id) or zero, and so taking the inverse limit does yield an exact sequence

$$0 \longrightarrow A^\flat \overset{\times \pi^\flat}{\longrightarrow} A^\flat \overset{\sharp \bmod \pi}{\longrightarrow} A/\pi A \longrightarrow 0.$$

That is, $\pi^\flat$ is a non-zero-divisor of $A^\flat$ and the untilting map induces an isomorphism of rings $A^\flat/\pi^\flat A^\flat \overset{\cong}{\to} A/\pi A$.

Finally we check that the topology on $A^\flat$ is the $\pi^\flat$-adic topology. Since $A^\flat \cong \varprojlim_{x\mapsto x^p} A/\pi A$ is a homeomorphism by part (i), a basis of open neighbourhoods of $0 \in A^\flat$ is given by $\mathrm{Ker}(\mathrm{proj_n})$ for $n \geq 0$, where $\mathrm{proj}_n : \varprojlim_{x\mapsto x^p} A/\pi A \to A/\pi A$, $(b_0, b_1, \dots) \mapsto b_n$ denotes the n^{th}-projection map. Note that proj_0 is the untilting map. Since the composition $A^\flat \overset{\varphi^n \cong}{\longrightarrow} A^\flat \overset{\mathrm{proj}_n}{\longrightarrow} A/\pi A$ is proj_0, the basis of open neighbourhoods is given by

$$\mathrm{Ker}(\mathrm{proj_n}) = \varphi^n(\mathrm{Ker}(\mathrm{proj}_0)) = \varphi^n(\pi^\flat A^\flat) = \pi^{\flat p^n} A^\flat,$$

showing that the topology is indeed $\pi^\flat$-adic. $\square$

We point out explicitly that we showed in the previous proof that the kernel of the surjective ring homomorphism $A^\flat \overset{\# \bmod \pi}{\longrightarrow} A/\pi A$ (i.e., projection to the 0^{th}-coordinate of $A^\flat = \varprojlim_{x\mapsto x^p} A/\pi A$) is $\pi^\flat A^\flat$. This will be used repeatedly.